REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY (Leave Blank)	2. REPORT DATE 8 February 2005	3. REPORT TYPE AND DATES COVERED Final report, June 1, 2004-Feb. 28, 2005

4. TITLE AND SUBTITLE
Physically nonlinear behavior of piezoelectric actuators subject to high electric fields

5. FUNDING NUMBERS
Contract W911NF-04-1-0192

6. AUTHOR(S)
Dr. Victor Birman

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Curators of the University of Missouri, 1870 Miner Circle,
OSP-215 ME Annex, Rolla, MO 65409-0970

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U. S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
45137.1-EG-II

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12 a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12 b. DISTRIBUTION CODE
.

13. ABSTRACT (Maximum 200 words)

The paper is concerned with a physically nonlinear piezoelectric material behavior and its applications to practical problems. A survey of work dealing with the phenomenon is included in the introduction. Subsequently, the emphasis is on the analysis of vibrations of piezoelectric rods where a rather unique situation is observed, i.e. the response of a nonlinear system can be modeled by linear equations of motion. The solutions are obtained analytically by the Lagrange equation and by the Generalized Galerkin procedure. Applying either of these methods, the study of forced vibrations of a physically nonlinear piezoelectric rod subject to a periodic electric field in the axial direction is reduced to the analysis of a system of nonhomogeneous Mathieu-Hill equations. In the particular case where the interaction between axial and radial vibrations can be neglected, the closed-form solution for the former vibrations is obtained in the paper and it is shown that both the Lagrange equation and the Generalized Galerkin procedure yield identical results. Numerical examples presented in the paper elucidate the significance of physically nonlinear effects that should not be arbitrary disregarded in design, without a proper evidence.

14. SUBJECT TERMS
Piezoelectric materials, actuators, nonlinear problems, physical nonlinearity, vibrations

15. NUMBER OF PAGES
42

16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	20. LIMITATION OF ABSTRACT	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	**UL**	**UNCLASSIFIED**	**UL**

Final Report

Physically Nonlinear Behavior of Piezoelectric Actuators Subject to High Electric Fields

Submitted to the US Army Research Office

Victor Birman
University of Missouri-Rolla
Engineering Education Center
8001 Natural Bridge Road
St. Louis, Missouri 63121

February 8, 2005

Table of Contents

The results of this research will be submitted to an archival journal for possible publication. Accordingly, most of the following sections of the report are prepared in the form of such paper.

The results will also be presented at IMECE-2005 (International Mechanical Engineering Congress) as a part of the symposium on adaptive structures (invited paper).

Details of these submissions will be forwarded to ARO as soon as the corresponding papers are accepted for publication and presentation.

Abstract

The paper is concerned with a physically nonlinear piezoelectric material behavior and its applications to practical problems. A survey of work dealing with the phenomenon is included in the introduction. Subsequently, the emphasis is on the analysis of vibrations of piezoelectric rods where a rather unique situation is observed, i.e. the response of a nonlinear system can be modeled by linear equations of motion. The solutions are obtained analytically by the Lagrange equation and by the Generalized Galerkin procedure. Applying either of these methods, the study of forced vibrations of a physically nonlinear piezoelectric rod subject to a periodic electric field in the axial direction is reduced to the analysis of a system of nonhomogeneous Mathieu-Hill equations. In the particular case where the interaction between axial and radial vibrations can be neglected, the closed-form solution for the former vibrations is obtained in the paper and it is shown that both the Lagrange equation and the Generalized Galerkin procedure yield identical results. Numerical examples presented in the paper elucidate the significance of physically nonlinear effects that should not be arbitrary disregarded in design, without a proper evidence.

Introduction

The well known physically linear relationships between the tensors of stress and strain and the vectors of electric field and electric displacement for piezoelectric materials are applicable to a particular case where nonlinear effects are negligible. While geometric nonlinearity can be partially incorporated into these equations by an appropriate choice of the strain-displacement relationships, neither this nonlinearity nor physical nonlinearity of the material are fully reflected in the linear version of the constitutive equations. A general form of constitutive equations accounting for nonlinear products of the components of the strain vector (geometrically nonlinear effect) and physically nonlinear terms has been derived and published (see for example, Maugin et al., 1992). However, the complexity of these equations as well as the lack of experimental data on the coefficients at the nonlinear terms and the difficulty involved in their evaluation prevented a wide acceptance of nonlinear constitutive equations in design and practical applications. This makes it important to elucidate a relative contribution of geometrically and physically nonlinear terms and to assess both the necessity of their incorporation in the analysis and their relative qualitative and quantitative effects on the solution.

Early studies of the effects of stress and electric fields on the response of piezoelectric materials were conducted by Berlington and Krueger (1959), Woolett and Leblanc (1973), Krueger (1954, 1967, 1968a, 1968b), Brown and McMahon, (1962, 1965) and Fritz (1978). A nonlinear nature of the problem is evident in these investigations. The theoretical formulation was also developed by Tiersten (1971) who later applied it to the problems of thin and membrane piezoelectric plates subjected to high electric fields (Tiersten, 1993a, b). Other derivations of physically nonlinear equations were published by Nakagawa et al. (1973) and Cho and Yamanouchi (1987).

Notably, a physically nonlinear behavior of piezoelectric materials is also reflected in the hysteresis or butterfly loops in the electric field – strain or strain-stress planes. Such loops and relevant nonlinear phenomena were described and discussed by a number of authors (Chen and Montgomery, 1980; Chen and Madsen, 1981; Bassiouny et al., 1988; Leigh and Zimmerman, 1991; Ge and Jouaneh, 1995, etc.)

Beige and Schmidt (1982) and Beige (1983) included higher-order electric and elastic terms in their studies of longitudinal vibrations of a plate with a 31 piezoelectric effect. This effect has also been studied by von Wagner and Hagedorn (2002) for piezoelectric beams. Other studies of von Wagner (2003, 2004) and von Wagner and Hagedorn (2003) were concerned with physical nonlinearity in 33-piezoelectrics. The work of Chattopadhyay et al. (1999) employed nonlinear constitutive equations incorporating a cubic nonlinearity for the electric field to analyze helicopter blades (modeled by composite box beams). The subsequent work accounted for transverse shear deformability of monocoque and sandwich plates combined with the nonlinear piezoelectric effect of embedded or mounted sensors and actuators (Thornburgh and Chattopadhyay, 2001).

Among recent studies that attempt to implicitly account for nonlinearities by using variable coefficients in linear constitutive relationships one can mention the paper by Sherritt et al. (1996) where the piezoelectric coefficient d_{33} of lead zirconate titanate ceramics was shown dependent on stress as well as being a function of temperature and frequency. In particular, a step stress caused a time dependent variation in this coefficient that was attributed to a slow movement of 90^o domain walls in the material. Further studies of the effects of various factors on the coefficients in linear constitutive equations for piezoelectrics were conducted by the same group at the Royal Military College of Canada. In particular, experimental data elucidating these phenomena was presented in the papers of Wiederick et al. (1996) and Sherrit et al. (1996, 1997). In these studies, it was shown that both the piezoelectric constants as well as the permittivity are nonlinear functions of the applied electric field. The experimental approach employed to measure piezoelectric constants was based on the so-called optical lever used to measure the strains in the 1 or 3 directions as functions of the applied electric field in the 3-direction. It was also observed in these studies that the piezoelectric coefficients increase almost linearly with the stress. The nonlinear contributions became much more pronounced at the applied electric field exceeding 500-1000 V/mm.

The intrinsic and extrinsic contributions to the piezoelectric effect were discussed by Yang et al. (2000). While the intrinsic property is typical for a single domain crystal, the contributions associated with the presence of multiple crystals are usually extrinsic. Such contributions are dominant in soft piezoceramics where they are associated with domain switching under the influence of high electric fields (Mukherjee et al., 2001). In general, it has been observed that piezoelectric coefficients are nonlinear functions of the applied compressive stress. As follows from this paper and from other publications of the same group (see for example, Ren et al., 2000), piezoelectric coefficients d_{31}, d_{33}, d_{15} typically increase with the electric field, particularly in soft piezoceramics, while an increase in the frequency results in a very small decrease of these coefficients.

Wang and Carman (1995) showed in their experiments that both the coefficient of thermal expansion as well as the piezoelectric coefficient d_{31} are affected by a cryogenic temperature. Moreover, it was shown that d_{31} is affected by the magnitude of strain. In addition, the electric field was recently shown to affect the elastic modulus (Chaplya and Carman, 2002).

Barrett (1995) showed a nonlinear relationship between the strain in the piezoelectric (PZT-5H) actuator and the applied electric field. The nonlinearity was observed, even though the electric field considered in this paper was limited to less than 500 V/mm.

Bert and Birman (1998) proved that both the stress and temperature affect the coefficient d_{31} in a one-dimensional problem. In addition, they characterized the variations of the coefficient of thermal expansion with the stress and electric field. For example, it was shown that CTE of PZT-4 increases by 10% if the stress reaches the static strength value, and even more remarkable, CTE of PZT-5A increased by 15.8% under the electric field equal to 2,000 V/mm. This work was further expanded by Bert and Birman (1999) to two-dimensional and three-dimensional cases.

Joshi (1992) derived physically nonlinear constitutive equations for piezoceramics by the assumption that material constants are independent of the magnitude of stress or electric field. This solution was obtained using the thermodynamic Gibbs potential and retaining the second-order terms in the total differentials of dependent variables (strains, electric flux density, and entropy). Numerical examples were not presented in this paper making it impossible to estimate a relative contribution of various terms in the physically nonlinear formulation. An important contribution was related to accounting for the electrostrictive and elastostrictive effects. The constitutive equations for a physically nonlinear material derived by Maugin et al. (1992) based on the analysis of the volume energy, i.e. the energy density of an electroelastic solid, are similar to equations of Joshi (1992).

A nonlinear relationship between the deflection of the tip of a bimorph working in the 31 mode and the applied electric field was observed for various piezoelectric actuators by Wang et al. (1999) who attributed this nonlinearity to an increase of d_{31} with the applied electric field. Note that the physical nonlinearity was reported in this paper, although the electric field was relatively low (150 V/mm).

The effect of physical nonlinearity on shape control of composite laminated beams was considered by Achuthan et al. (2001). As follows from this study, the voltage required to control the shape of the beam is significantly reduced when physical nonlinearity of piezoelectric patches on the beam surface is taken in to account.

Notably, the electrostrictive response can include higher-order nonlinear terms, in addition to the well-known quadratic relationship between the stress and the electric field

(Sherrit et al., 1999). However, these nonlinear contributions become essential only in the case of very high applied fields.

Explicit expressions for piezoelectric coefficients d_{31}, d_{33} as functions of the peak-to-peak voltage amplitude were obtained for a class of materials by Williams (2004). These relationships involved quadratic power of the peak-to-peak voltage, reflecting physical nonlinearity.

Representative empirical equations including physical nonlinearity are shown below. For example, the results communicated by Barrett (2002) regarding the response of G-1195 piezoceramics under the electric field of up to 600 V/mm include the following nonlinear relationship between the microstrain and the electric field (31 effect):

$$\varepsilon = \left(0.227E_z + 0.000243E_z^2\right)*10^{-6} \tag{1}$$

Priya et al. (2001) found that the relationship between the coefficient d_{31} and the squared applied elastic strain is linear. In particular, for soft PZT-5A, this relationship was obtained in the form

$$d_{31} = 2.03*10^{-10} + 1.12*10^{-4}\varepsilon^2 \tag{2}$$

where the squared strain varied from zero to $2.8*10^{-7}$ and the piezoelectric coefficient was measured in C/N.

The present paper concentrates on the investigation of physically nonlinear effects on the behavior of piezoelectric rods polarized in the axial direction. Such problems are important in piezoelectric transducers (Fig. 1) used in underwater hydrophones, acoustic imaging and medical applications. Typically, the behavior of piezoelectric rods and 1-3 piezocomposites consisting of rods embedded in the matrix are studied using linear constitutive equations (Li and Sottos, 1995, 1996a,b, Sigmund et al., 1998). Recent papers by Tan and Tong (2001, 2002) extended the study to the physically nonlinear static formulation. Physically nonlinear dynamic problems were also considered by Wagner (2003, 2004) and Wagner and Hagedorn (2003). A related problem of linear vibrations of thin piezoceramic discs accounting for coupled axial, tangential and radial modes was analyzed by Huang et al (2004). In the present paper, a unified formulation is presented enabling us to identify essential physically nonlinear effects and conduct a comprehensive analysis of the problem. As follows from this paper, physically nonlinear effects may significantly influence the dynamic response of a piezoelectric rod.

Analysis

1. Derivation of physically nonlinear constitutive relationships for an orthotropic cylindrical piezoelectric rod subject to an electric field in the axial direction

Consider a cylindrical rod shown in Fig. 2 subject to an electric field in the z-direction. For convenience, the following notations employed in the subsequent discussion are introduced at this phase. The cylindrical coordinate axes z, r, θ identified in Fig. 1 are also denoted as 3, 1 and 2, respectively. The problem being axisymmetric, the circumferential displacement is equal to zero, while the axial and radial displacements are denoted by w and u, respectively.

The rod subject to a dynamic excitation induced by an electric field $E_z = E_3$ experiences vibrations in the axial and radial directions, while tangential motion occurs only if axisymmetry is violated by adjacent structures. If the side surface of the rod is not attached to other structural elements of if the adjacent structure is quasi-isotropic, the rod retains a cylindrical shape during these vibrations. Therefore, shearing stresses and strains are equal to zero. In the present analysis, the rod is assumed ‘anchored” at the plane $z = 0$, preventing axial displacements at this location.

The constitutive relations are derived following the approach by Maugin et al. (1992). Note that notations used in this book differ from a number of references where similar relationships were derived, such as Joshi (1992). However, accounting for this difference does not result in a physically different formulation.

The energy density obtained neglecting geometric nonlinearities is (Maugin et al., 1992):

$$\Psi = \frac{1}{2}C_{\alpha\beta}\varepsilon_\alpha\varepsilon_\beta - e_{m\alpha}E_m\varepsilon_\alpha - \frac{1}{2}e_{m\alpha\beta}E_m\varepsilon_\alpha\varepsilon_\beta - \frac{1}{2}\varepsilon_{mn}E_mE_n - \frac{1}{6}\varepsilon_{mnp}E_mE_nE_p - \frac{1}{2}l_{mn\alpha}E_mE_n\varepsilon_\alpha \tag{3}$$

where

$C_{\alpha\beta}$ are elastic stiffness constants;

ε_α are strains;

$e_{m\alpha}$ are piezoelectric constants;

E_m are components of electric field in the corresponding direction;

$e_{m\alpha\beta}$ are electroelastic constants;

ε_{mn} and ε_{mnp} are dielectric coefficients (permittivity and third-order dielectric coefficients, respectively);

$l_{mn\alpha}$ are electrostrictive coefficients.

The constitutive relations yielding stresses and electric displacements are obtained as

$$\sigma_\alpha = \frac{\partial\Psi}{\partial\varepsilon_\alpha}, \qquad D_m = -\frac{\partial\Psi}{\partial E_m} \tag{4}$$

This yields the following expressions for the components of the stress tensor and electric displacements:

$$\sigma_\alpha = \underline{C_{\alpha\beta}\varepsilon_\beta - e_{m\alpha}E_m} - e_{m\alpha\beta}E_m\varepsilon_\beta - \frac{1}{2}l_{mn\alpha}E_mE_n \tag{5}$$

$$D_m = \underline{e_{m\alpha}\varepsilon_\alpha} + \frac{1}{2}e_{m\alpha\beta}\varepsilon_\alpha\varepsilon_\beta + \underline{\varepsilon_{mn}E_n} + \frac{1}{2}\varepsilon_{mnp}E_nE_p + l_{mn\alpha}E_n\varepsilon_\alpha \tag{6}$$

The terms underlined in the right side of these equations are retained in the linear formulation. As indicated above, numerous authors implicitly account for the nonlinearity by using variable coefficients of these terms, dependent on various factors. Note that the quadratic function of strain in (6) can usually be neglected even if the problem is geometrically nonlinear.

In the problem considered in this paper, we are concerned with a dynamic response of the rod to the applied field in the axial direction, i.e. $E_3(t) = E_z(t)$. Accordingly, the nonzero elements of the tensor of stress given by (5) are

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \sigma_z\end{Bmatrix} = \begin{bmatrix} C_{11}-e_{311}E_3 & C_{12}-e_{312}E_3 & C_{13}-e_{313}E_3\\ C_{12}-e_{321}E_3 & C_{22}-e_{322}E_3 & C_{23}-e_{323}E_3\\ C_{13}-e_{331}E_3 & C_{23}-e_{332}E_3 & C_{33}-e_{333}E_3\end{bmatrix}\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\\ \varepsilon_z\end{Bmatrix} - \begin{Bmatrix}e_{31}\\ e_{32}\\ e_{33}\end{Bmatrix}E_3 - \frac{1}{2}\begin{Bmatrix}l_{331}\\ l_{332}\\ l_{333}\end{Bmatrix}E_3^2 \tag{7}$$

2. Analysis of uncoupled axial vibrations

This section presents the solution of the problem of uncoupled axial vibrations of a piezoelectric rod activated by a periodic in time electric field in the axial (z) direction. The solution is obtained by assumption that the effect of radial vibrations on the axial motion can be disregarded.

Two methods of the analysis are considered, namely the Generalized Galerkin procedure and the Lagrange equation. In particular, the former procedure is different from the conventional and well known Galerkin procedure in enabling one to address the issue of boundary conditions that cannot be satisfied by expressions chosen to represent displacements. Although this procedure is relatively little known, the solutions utilizing it have been published (Simitses, 1986; Houbolt and Brooks, 1958). The choice of the appropriate sign for the term incorporating the boundary conditions into the Generalized Galerkin formulation can only be established if the equations of equilibrium and the boundary conditions are derived from the Hamilton principle. Accordingly, the analysis begins with the consideration of this principle in application to the present problem.

2.1. Derivation of the equation employed in the Generalized Galerkin procedure from the Hamilton principle

The Hamilton principle applied to the problem of axial vibrations of a piezoelectric rod where the motion is excited by an electric field and external mechanical forces are absent is formulated as

$$\int_{t_0}^{t_1} (\delta V - \delta T) dt = 0 \tag{8}$$

where the integration is conducted over an arbitrary interval of time $t_0 < t < t_1$, V is the strain energy, and T is the kinetic energy.

The strain energy accumulated in the rod that experiences vibrations in the axial direction is given by

$$\delta V = \int_0^{2\pi} \int_0^{a} \int_0^{h} \sigma_z \delta \varepsilon_z r dz dr d\theta \tag{9}$$

where a is the radius of the rod, and the variation of the axial strain is related to the variation of the axial displacement w by

$$\delta \varepsilon_z = (\delta w)_{,z} \tag{10}$$

The substitution of (10) into (9) and the integration by parts yields

$$\delta V = \pi a^2 \int_{t_0}^{t_1} \left[(\sigma_z \delta w)_{z=h} - (\sigma_z \delta w)_{z=0} - \int_0^h (\sigma_z)_{,z} \, \delta w dz \right] dt \tag{11}$$

The variation of the kinetic energy is

$$\delta T = \pi a^2 m \int_0^h \dot{w} \delta \dot{w} dz \tag{12}$$

where m is the mass density of the rod material.

Integrating this expression by parts one obtains

$$\delta T = \pi a^2 m \int_0^h \left[(\dot{w} \delta w)_{t=t_1} - (\dot{w} \delta w)_{t=t_0} - \int_{t_0}^{t_1} \ddot{w} \delta w dt \right] dz \tag{13}$$

The first two terms under the integral in the right side of (13) can be taken equal to zero assuming that $\delta w(t = t_1) = \delta w(t = t_0) = 0$ (Whitney, 1987). Then substituting the variations of strain and kinetic energies given by (11) and (13), respectively, into (8) one obtains both the equation of motion as well as the boundary conditions that are actually well known (the reason for the previous derivation is explained below):

$$m\ddot{w} - \sigma_z,_z = 0 \tag{14}$$

$$\begin{aligned} z = 0: \qquad & \sigma_z = 0 \;\; or \;\; \delta w = 0 \\ z = h: \qquad & \sigma_z = 0 \;\; or \;\; \delta w = 0 \end{aligned} \tag{15}$$

The boundary conditions that can be specified from (15) reflect that in the present problem the cross section $z = 0$ is prevented from axial motion, so that $\delta w(z = 0) = 0$. The axial stresses at the free end of the rod should be equal to zero, i.e. $\sigma_z(z = h) = 0$.

In the case where it is impossible to satisfy the stress boundary conditions, following the Generalized Galerkin procedure, the displacements can be sought in the form

$$w(z,t) = \sum_i Z_i(z) T_i(t) \tag{16}$$

and the system of equations of motion with respect to the functions of time $T_i(t)$ available from (8) by substituting (11) and (13) is

$$\int_0^h (\sigma_z,_z - m\ddot{w}) Z_i(z) dz - \sigma_z(z = h) Z_i(h) = 0 \tag{17}$$

It is observed that the sign of the second term in (17) would not be evident without the previous derivation.

2.2 Analysis using normal modes to represent the motion

The motion considered in this section is represented in terms of normal modes of axial vibrations of the rod. Two methods employed to analyze the problem are the Generalized Galerkin procedure and the Rayleigh-Ritz method. As is shown below, these methods yield identical systems of equations of motion.

The axial free vibrations of a rod with the boundary conditions specified above are represented in the form

$$w(x,t) = \sum_i \left(A_{1i} \sin \lambda_i t + A_{2i} \cos \lambda_i t\right) \sin \frac{i\pi z}{2h} \tag{18}$$

where i is a natural odd number, A_{ki} are constants of integration specified from the initial conditions, and $\lambda_i = \frac{i\pi}{2h}\sqrt{\frac{C_{33}}{m}}$ is a natural frequency.

2.2.1. Generalized Galerkin procedure

Following the Galerkin procedure, the axial displacements should be chosen in the form of series satisfying the boundary conditions. However, it appears impossible to choose such series, while satisfying the conditions both at the "anchored" cross section $z = 0$ as well as at the free end of the rod $z = h$. Therefore, the solution is sought in the form

$$w = \sum_i W_i(t) h \sin \frac{i\pi z}{2h} \tag{19}$$

The substitution of (19) into (17) where $Z_i = \sin \frac{i\pi z}{2h}$ yields a system of uncoupled equations. In particular, the i-th equation of this system is

$$\frac{mh^2}{2}\ddot{W}_i + \left(C_{33} - e_{333}E_z\right)\frac{i^2\pi^2}{8}W_i = \left(e_{33}E_z + \frac{1}{2}l_{333}E_z^2\right)\sin\frac{i\pi}{2} \tag{20}$$

2.2.2. Solution by the Lagrange equation

The energy density within a rod vibrating in the axial direction can be obtained from (3):

$$\Psi' = \frac{1}{2}C_{33}\varepsilon_z{}^2 - e_{33}E_z\varepsilon_z - \frac{1}{2}e_{333}E_z\varepsilon_z{}^2 - \underline{\frac{1}{2}\varepsilon_{33}E_z{}^2 - \frac{1}{6}\varepsilon_{333}E_z{}^3} - \frac{1}{2}l_{333}E_z{}^2\varepsilon_z \tag{21}$$

Note that the terms dependent only on the electric field that are underlined in (21) do not affect the subsequent solution and are omitted.

If axial vibrations of the rod are represented by series (19), the Lagrange equation for the n-th term of these series is

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{W}_n}\right) - \frac{\partial T}{\partial W_n} + \frac{\partial V}{\partial W_n} = 0 \tag{22}$$

where the kinetic and strain energy contributions are given by

$$T = \frac{1}{2}\int_0^{2\pi}\int_0^h\int_0^a m\dot{w}^2 r\,dr\,dz\,d\theta \tag{23}$$

and

$$V = \int_0^{2\pi}\int_0^h\int_0^a \Psi' r\,dr\,dz\,d\theta \tag{24}$$

respectively.

The substitution of (19) into (21, 23, and 24) and the subsequent application of the Lagrange equation (22) yields the same system of equations of motion as that obtained by the Generalized Galerkin procedure, i.e. (20).

2.3 Analysis using power series to represent the motion

The solution can also be sought in the power series, i.e.

$$w = \sum_{j=1} \overline{W}_j(t) z^j \tag{25}$$

where $W_j(t)$ are unknown functions of time (j are natural numbers that can be both even and odd).

2.3.1. Generalized Galerkin procedure

It is immediately obvious that series (25) satisfy the boundary condition $w(z=0)=0$, while the condition of zero axial stress at the free end is violated. Accordingly, the Generalized Galerkin procedure implies that the n-th equation of the system of equations of motion is

$$\int_0^h (\sigma_z,_z - m\ddot{w}) z^n dz - \sigma_z(z=h) h^n = 0 \tag{26}$$

Note that (26) was written accounting for the axisymmetry of the problem and accordingly, avoiding the integration in the circumferential direction.

The substitution of the axial stress from (7) and axial displacements presented by (25) into (26) and using the strain-displacement relationship $\varepsilon_z = w,_z$ yields the system of coupled equations of motion. The n-th equation is

$$m\sum_{j=1} \frac{h^{j+n+1}}{j+n+1} \ddot{\overline{W}}_j + (C_{33} - e_{333}E_z)\left(h^n \overline{W_1} + n\sum_{j=2} \frac{jh^{j+n-1}}{j+n-1} \overline{W}_j \right) = \left(e_{ee}E_z + \frac{1}{2} l_{333} E_z^2 \right) h^n \tag{27}$$

2.3.2. Solution by the Lagrange equation

The energy density within a rod vibrating in the axial direction is given by (21). If axial vibrations of the rod are represented by the power series (25), the Lagrange equation yields the following system of equations

$$m\sum_{j=1} \frac{h^{j+n+1}}{j+n+1} \ddot{\overline{W}}_j + (C_{33} - e_{333}E_z) n \sum_{j=1} j \frac{h^{j+n-1}}{j+n-1} \overline{W}_j = \left(e_{33}E_z + \frac{1}{2} l_{333} E_z^2 \right) h^n$$

(28)

It is easy to show that the systems of equations (27) and (28) are identical, i.e. the Generalized Galerkin procedure and the Lagrange equation yield identical equations of motion, similar to the case where the mode shape of vibration was represented by normal modes.

2.4. Solution of equations of axial motion

The systems of equations (20) and (28) can be integrated if the electric field is a known function of time. In the present work it is assumed that this function is periodic, i.e.

$$E_z(t) = E\cos\omega t \tag{29}$$

Then the substitution of (29) into one of the above-mentioned systems of equations yields a system of equations of motion that can be solved either analytically or numerically. For example, the system (28) becomes

$$m\sum_{j=1}\frac{h^{j+n+1}}{j+n+1}\ddot{\overline{W}}_j + (C_{33} - e_{333}E\cos\omega t)n\sum_{j=1} i\frac{h^{j+n-1}}{j+n-1}\overline{W}_j = \left(\frac{1}{4}l_{333}E^2 + e_{33}E\cos\omega t + \frac{1}{4}l_{333}E^2\cos 2\omega t\right)h^n \tag{30}$$

The substitution of (29) into (20) yields

$$\frac{mh^2}{2}\ddot{W}_i + (C_{33} - e_{333}E\cos\omega t)\frac{i^2\pi^2}{8}W_i = \left(\frac{1}{4}l_{333}E^2 + e_{33}E\cos\omega t + \frac{1}{4}l_{333}E^2\cos 2\omega t\right)\sin\frac{i\pi}{2} \tag{31}$$

The systems of equations (30) and (31) represent systems of nonhomogeneous Mathieu-Hill equations. Such equations were encountered since the first part of the last century (Strutt, 1932; McLachlan, 1947) when studies were often limited to homogenous equations that are employed to solve problems of dynamic or parametric stability of structures (Bolotin, 1964). A typical problem where the analysis is reduced to a nonhomogeneous Mathieu equation is the motion of a rod with initial imperfection subject to a periodic in time axial force (Bolotin, 1964). A related problem is that of forced vibrations of a rod subject to an eccentrically applied driving force. In addition to the classical monograph of Bolotin (1964), a number of investigations have been concerned with the problem of an interaction between forced and parametric vibrations (Hsu and Cheng, 1974; Nguyen, 1975; Troger and Hsu, 1977; HaQuang et al., 1987a,b; Plaut et al., 1990; Nguyen and Ginsberg, 2001).

The analytical solution of the system of equations (30) can be sought in the form of trigonometric time series:

$$\overline{W}_j = A_{0j} + \sum_r A_{rj} \cos\frac{r\omega t}{2} + \sum_p B_{pj} \sin\frac{p\omega t}{2} \tag{32}$$

where A_{0j}, A_{rj} and B_{pj} are unknown coefficients. These coefficients can be determined by substituting the series (32) into (30) or (31), equating the coefficients at the same trigonometric functions of time and solving the system of resulting linear algebraic equations with respect to A_{0j}, A_{rj} and B_{pj}. The solution of (31) can be obtained by the same approach. Details of the straightforward solution of the system of linear equations with respect to A_{0j}, A_{rj} and B_{pj} are omitted here for brevity.

If the axial displacement is assumed to be a linear function of the distance from the cross section $z = 0$, i.e. $n = j = 1$, and the electric field is given by (29), the system (30) is reduced to a single equation

$$\frac{mh^2}{3}\ddot{\overline{W}}_1 + C_{33}\overline{W}_1 - e_{333}E\overline{W}_1 \cos\omega t = \frac{1}{4}l_{333}E^2 + e_{33}E\cos\omega t + \frac{1}{4}l_{333}E^2\cos 2\omega t \tag{33}$$

The corresponding equation for the solution representing the motion in terms of normal modes is available from (31):

$$\frac{mh^2}{2}\ddot{W}_1 + \left(C_{33} - e_{333}E\cos\omega t\right)\frac{\pi^2}{8}W_1 = \frac{1}{4}l_{333}E^2 + e_{33}E\cos\omega t + \frac{1}{4}l_{333}E^2\cos 2\omega t \tag{34}$$

As is shown below, single-term solutions that result in equations (33) and (34) are always accurate if the driving frequency ω is smaller than the fundamental frequency of the rod.

The systems of equations (30) and (31) or equations (33) and (34) could be simplified in case of numerous piezoelectric materials as shown below. It can be observed that physical nonlinearity in the expression for the energy density as given by (21) results in terms proportional to e_{333} and l_{333}. Consider for example, equation (34) where it is instructive to compare the magnitude of the following coefficients:

$$\begin{aligned} k_1 &= e_{333}EW_1 \\ k_2 &= \frac{1}{4}l_{333}E^2 \\ k_3 &= e_{33}E \end{aligned} \tag{35}$$

Note that a physically linear formulation can be obtained by setting $k_1 = k_2 = 0$.

Two materials chosen for the following comparison and considered in numerical examples are PZT-5H and PZN-4.5%PT (Tan and Tong, 2001, 2002). In addition, $LiNbO_3$ (Maugin, et al., 1992) is used in the analysis of the coefficients (35). The magnitude of typical electric fields usually varies from zero to 2.0 MV/m (Maugin et al., 1992). The values of material constants needed to evaluate the coefficients in (35) are listed in Table 1.

Table 1. Material constant of representative piezoelectric materials

Material	e_{33} $\left(C/m^2 \right)$	e_{333} $\left(C/m^2 \right)$	C_{33} $\left(GPa \right)$	l_{333} $\left(F/m \right)$
PZT-5H	24.54 (or 31.02)	$5.7*10^4$	108.0	$-21.21*10^{-6}$
PZN-4.5%PT	12.1 (or 13.02)	$4.2*10^4$	89.0	$-1.61*10^{-6}$
$LiNbO_3$	1.3	-17.3	24.5	$-2.76*10^{-9}$

Using a high electric field, i.e. 2.0 MV/m, since it results in a stronger nonlinear effect, one obtains the values of the coefficients in (35) listed in Table 2.

Table 2. Coefficients of the equations of motion for representative materials

Material	k_1	k_2	k_3
PZT-5H	$11.4*10^{10}W_1$	$-2.12*10^7$	$5.09*10^7$
PZN-4.5%PT	$8.9*10^{10}W_1$	$-1.61*10^6$	$2.42*10^7$
$LiNbO_3$	$-34.6*10^6W_1$	$-2.76*10^3$	$2.6*10^6$

Note that W_1 in Table 1 is nondimensional and the units of all coefficients are $C*V/m^3$. A comparison of the coefficients in Table 2 yields the conclusion that the terms proportional to the squared electric field, i.e. k_2, can be neglected if the electric field remains within certain limits. In particular, these terms can be neglected for PZN-4.5%PT and for $LiNbO_3$ even at 2.0MV/m, while the term proportional to k_2 can be neglected for PZT-5H if the field remains smaller than 0.5MV/m since at this electric field k_2 is an order of magnitude higher than k_3. Accordingly, if the electric field is below the limits specified above, the systems of equation (30) and (31) as well as equations (33) and (34) can be simplified. In particular, the latter equations are reduced to a nonhomogeneous Mathieu equation:

$$\ddot{W}_1 + \lambda^2 W_1 - 2\mu\lambda^2 W_1 \cos\omega t = p\cos\omega t \qquad (36)$$

where the fundamental frequency, the parametric loading coefficient and the forcing function are

$$\begin{aligned} &\lambda_p = \frac{1}{h}\sqrt{\frac{3C_{33}}{m}} \qquad \mu_p = \frac{e_{333}E}{2C_{33}} \qquad p_p = \frac{3e_{33}E}{mh^2} \qquad for\ (33) \\ &\lambda_n = \frac{\pi}{2h}\sqrt{\frac{C_{33}}{m}} \qquad \mu_n = \frac{e_{333}E}{2C_{33}} \qquad p_n = \frac{2e_{33}E}{mh^2} \qquad for\ (34) \end{aligned} \qquad (37)$$

where the subscripts identify the solutions in power series (p) and in normal modes (n).

As will be shown below, viscous damping is essential in the case where the driving frequency is close to the fundamental frequency of the rod. The corresponding expansion of (36) is

$$\ddot{W}_1 + \beta W_1 + \lambda^2 W_1 - 2\mu\lambda^2 W_1 \cos\omega t = p\cos\omega t \tag{38}$$

where $\beta = c/m$, c being a damping coefficient (for both eqns. 33 and 34). The value of the damping coefficient can be evaluated from published data for a quality factor Q as

$$c = 2c_{cr}Q \tag{39}$$

where c_{cr} is a critical damping coefficient of the rod experiencing axial vibrations.

Equations (36) or (38) are similar to the equation for lateral vibrations of an imperfect rod subjected to a periodic in time axial force (Mettler, 1941; Bolotin, 1964). Following these references, the solution for the steady state vibrations in the vicinity of the secondary region of parametric instability, i.e. in the case where the frequency of the electric field is close to the fundamental frequency can be adequately predicted retaining only three terms in series (32):

$$W_1 = A_0 + A_2\cos\omega t + B_2\sin\omega t \tag{40}$$

Using (40), the ratio of the amplitude of vibrations neglecting physical nonlinearity, i.e., using $\mu = 0$ in (36), to the corresponding amplitude accounting for the physically nonlinear effect is obtained in the form

$$R = 1 - \frac{2\mu^2}{1 - \left(\omega/\lambda\right)^2} \tag{41}$$

In the presence of viscous damping, this ratio is

$$R = 1 - \frac{2\mu^2}{1 - \left(\omega/\lambda\right)^2 - \dfrac{\beta^2\left(\omega/\lambda\right)^2}{\lambda^2\left[\left(\omega/\lambda\right)^2 - 1\right]}} \tag{42}$$

3. Solution for coupled axial-radial axisymmetric vibrations by the Generalized Galerkin procedure

Equations of the axisymmetric motion of the rod are

$$
\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = m\ddot{u} \\
&\frac{\partial \sigma_z}{\partial z} = m\ddot{w}
\end{aligned} \tag{43}
$$

where the terms in the right side represent inertias in the radial and axial directions.

The solution must satisfy the following conditions:

$$
\begin{aligned}
&r = 0: \qquad && u = 0 \\
&z = 0: && w = 0 \\
&r = a: && \sigma_r = 0 \\
&z = h: && \sigma_z = 0
\end{aligned} \tag{44}
$$

It is evident that the exact satisfaction of the static (stress) conditions in (44) is impossible.

The following approach to the solution utilizes the Generalized Galerkin procedure. For example, the displacements can be sought in the form of power series that satisfy the first two conditions (44):

$$
u = \sum_{j=1} U_j(t) r^j \qquad\qquad w = \sum_{j=1} W_j(t) z^j \tag{45}
$$

The Generalized Galerkin procedure yields a set of equations. In particular, the n-th equation of motion in the radial direction (j=n) is

$$
\int_0^{2\pi}\int_0^h\int_0^a \left(\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} - m\ddot{u} \right) r^{n+1} dr dz d\theta - \int_0^{2\pi}\int_0^h \sigma_r (r = a) a^{n+1} dz d\theta = 0 \tag{46}
$$

The n-th equation of the axial motion obtained by this method becomes

$$
\int_0^{2\pi}\int_0^h\int_0^a \left(\frac{\partial \sigma_z}{\partial z} - m\ddot{w} \right) r z^n dr dz d\theta - \int_0^{2\pi}\int_0^a \sigma_z (z = h) r h^n dr d\theta = 0 \tag{47}
$$

The substitution of the series (45) into the constitutive relations (7) and the subsequent use of the Generalized Galerkin procedure (46) and (47) yield a system of coupled time-dependent differential equations for $U_j(t)$ and $W_j(t)$.

In the case where the solution is sought using normal modes of motion as generalized coordinates, series (45) are replaced with

$$u = \sum_{i=1} U_i(t) \sin\frac{i\pi r}{2a} \qquad w = \sum_{i=1} W_i(t) \sin\frac{i\pi z}{2h} \tag{48}$$

where i is an odd number. In this case, the Generalized Galerkin procedure implies

$$\int_0^{2\pi}\int_0^h\int_0^a \left(\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} - m\ddot{u}\right) r \sin\frac{n\pi r}{2a} drdzd\theta - \int_0^{2\pi}\int_0^h \sigma_r(r = a) a \sin\frac{n\pi}{2} dzd\theta = 0 \tag{49}$$

and

$$\int_0^{2\pi}\int_0^h\int_0^a \left(\frac{\partial\sigma_z}{\partial z} - m\ddot{w}\right) r \sin\frac{n\pi z}{2h} drdzd\theta - \int_0^{2\pi}\int_0^a \sigma_z(z = h) r \sin\frac{n\pi}{2} drd\theta = 0 \tag{50}$$

If the driving electric field is a periodic and harmonic function of time as is assumed in this paper (see Eqn. 29), the solution of the system of time-dependent equations of motion available from (46), (47) or from (48), (49) can be sought in the form of series

$$\begin{Bmatrix} U_n \\ W_n \end{Bmatrix} = \begin{Bmatrix} U_{n0} \\ W_{no} \end{Bmatrix} + \sum_{k=1,2,3,..} \begin{Bmatrix} U'_{nk} \sin\frac{k\omega t}{2} + U''_{nk} \cos\frac{k\omega t}{2} \\ W'_{nk} \sin\frac{k\omega t}{2} + W''_{nk} \cos\frac{k\omega t}{2} \end{Bmatrix} \tag{51}$$

Numerical results and discussion

The following results were obtained for two representative materials, i.e. PZT-5H and PZN-4.5%PT, using data from Table 1 (the values of e_{33} shown without brackets were employed in calculations). In the examples presented below the amplitude of the electric field was chosen equal to 2.0MV/m and 0.5MV/m for PZN-4.5%PT and PZT-5H, respectively. Accordingly, the terms proportional to the coefficient k_2 could be neglected. Notably, even in the case of a relatively low electric field for PZT-5H the effect of physical nonlinearity was significant as is shown in the following examples.

The effect of physical nonlinearity on the accuracy of the prediction of vibrations of a piezoelectric rod subject to electric fields with the driving frequency close to the fundamental frequency of the rod is illustrated in Figs. 3 and 4. The horizontal axes in these figures represent $F = \omega/\lambda$, so that in the absence of damping the amplitude ratio becomes infinite at the resonant frequency according to (41). As follows from Figs. 3 and 4, neglecting physical nonlinearity may cause a significant numerical error in the vicinity of the fundamental frequency, particularly at high electric fields. However, even at a relatively low electric field (Fig. 3), quantitative differences between physically nonlinear and linear solutions cannot be disregarded if $F \to 1$. It is noted that each of the curves in Figs. 3 and 4 has extreme values $R = 0$ and $R = \infty$, but some of these values could not be reflected in the figures due to a difficulty in graphing the results for the corresponding pinpointed values of F. Nevertheless, all essential trends are accurately identified in these figures.

The effect of damping on the ratio of amplitudes of linear and nonlinear vibrations is elucidated in Figs. 5 and 6. The quality factor employed to generate these results was chosen based on published data for PZT-5H. This factor was also applied to PZN-4.5%PT since damping data for this material is not available. However, independent of the exact damping coefficient value, it is evident that the qualitative effect of damping is present only in the immediate vicinity of the fundamental frequency (the same conclusion is valid for other resonant frequencies, though the effect becomes weaker at higher modes of motion). The effect of damping on vibrations of physically nonlinear piezoelectric rods is further elucidated in Figs. 7 and 8. As follows from these figures, the amplitudes of vibration are almost unaffected by damping outside a very narrow spectrum of driving frequencies encompassing the fundamental frequency. However, at the resonant frequency damping reduces the ratio of the amplitude of nonlinear vibrations with damping to that without damping (denoted RD in Figs. 7 and 8) to zero. This is anticipated since the amplitude of undamped motion is infinite at the resonance.

The previous results were generated using the normal mode approach. Therefore, it is important to compare the solutions obtained by this method to the results generated by the power series approach. Such comparison is shown in Figs. 9 and 10 where the ratio Rpq represents a ratio of the amplitudes of nonlinear vibrations obtained by the power series and normal modes approaches. The interpretation of the results for nonlinear vibrations shown in these figures should account for the fact that two driving frequencies, one of them yielding a ratio Rpn=0 and the second corresponding to an infinite Rpn reflect the resonances of the rod by the corresponding solutions. Naturally, the ratios Rpn exhibit abrupt variations within the resonance region. Outside the resonance region, the solutions are reasonably close to each other but the power series approach requires retention of more terms to accurately represent the motion.

The limits of the accuracy of a one-term solution by the normal mode method are elucidated in Figs. 11 and 12. As follows from these figures, the contribution of higher modes is negligible if the driving frequency is smaller than the fundamental frequency of the rod. As the driving frequency increases, the contribution of the second normal mode becomes essential (it was shown that higher modes have a negligible effect in the range of frequencies considered in Figs. 11 and 12). The increase of the contribution of the second mode is particularly pronounced in case of a high electric field.

Finally, the multi-mode solutions for the nondimensional amplitude of physically nonlinear axial vibrations of piezoelectric rods within a broad range of driving frequencies are illustrated in Figs. 13, 14 and 15. As is shown in these figures, the amplitudes of motion remain quite small, except for very narrow regions encompassing the resonant frequencies. However, though geometric nonlinearity can be neglected (with a possible exception of the above-mentioned regions), the physical nonlinearity should be accounted for as follows from the previous discussion.

Conclusions

The effect of physical nonlinearity on the dynamic response of piezoelectric rods polarized in the axial direction and subject to a harmonic electric field acting in this direction has been investigated. Such problems are important in a number of applications employing piezoelectric transducers. It is shown that physically nonlinear effects become more pronounced at higher fields. Mathematically, the presence of physical nonlinearity results in additional terms in the system of equations of motion, so that these equations change from the equations for forced vibrations in the physically linear case to the mixed forced-parametric dynamic formulation (Mathieu-Hill equations).

Physically nonlinear vibrations considered in the paper were investigated using the Generalized Galerkin procedure and the Lagrange equation. The former formulation enabling us to incorporate all boundary conditions was derived from the Hamilton principle. The solution was developed using two different systems of generalized coordinates, presenting the motion in terms of normal modes of free vibrations and in power series of the coordinates. Remarkably, in each of these cases, the Generalized Galerkin procedure and the Lagrange equation yield identical equations of motion. The solution considered in this paper was confined to uncoupled axial vibrations. However, the approach to the solution of a coupled axial-radial axisymmertic vibration problem has also been outlined.

Vibrations of the rods driven by a harmonic electric field were numerically investigated in the vicinity of the fundamental frequency. As follows from the representative examples, the error due to neglecting physical nonlinearity may be significant. This error increases in the close vicinity to the resonant frequency. The inaccuracy of the linear analysis becomes larger at higher electric fields.

Damping has a noticeable effect on vibrations of piezoelectric rods in the vicinity of the resonant frequency. However, the frequency range where the effect of damping is significant is very narrow. Within this range, damping results in a finite amplitude of vibrations, while without damping, the amplitude is infinite, similarly to the situation encountered in the problem of forced vibrations.

The analysis of vibrations based on using normal modes as generalized coordinates illustrates that higher modes have a negligible effect at the driving frequencies that are smaller than or close to the fundamental frequency of the rod. If the driving frequency increases beyond the fundamental frequency, the effect of higher modes becomes more pronounced, particularly at high electric fields. Therefore, a single-mode analysis can be safely employed only if the driving frequency remains below or close to the fundamental frequency, while a multi-mode analysis is recommended at higher values of the former frequency. The multi-mode analysis conducted for axial vibrations in the range of driving frequencies including the first two natural frequencies illustrated the presence of clearly identified peaks in the vicinity of the fundamental and second resonant frequencies. Outside these peaks, the amplitude of motion remained small, even at high electric fields.

A comparison between the results generated using normal modes and the solution employing power series to model the motion illustrated that these two solutions predict a qualitatively similar behavior. The exception was found in the vicinity of resonant frequencies where the solutions, while qualitatively similar, may predict significantly different amplitudes at the same driving frequency. In general, it is anticipated that the normal mode solution is more accurate, but as the number of terms in the series representing the motion increases, the difference between the solutions should become smaller.

Note that the effect of physical nonlinearity was also investigated as a part of the program on the development of Thunder actuators for QorTek, Inc. for control surfaces of minituarized munitions conducted by Dr. Birman. The position of the control surface was changed by activating piezoelectric layers that caused an appropriate bending of this surface. As was shown analytically (and confirmed in experiments), accounting for physical nonlinearity of piezoelectric actuator-layers resulted in a significant different response compared to a simplified approach neglecting such phenomenon. The comparison between physically nonlinear and experimental results for representative control surfaces was favorable (see details in Appendix).

References

Achuthan, A., Keng, A.K., and Ming, W.C., 2001. Shape control of coupled nonlinear piezoelectric beams. *Smart Materials and Structures*, **10**, pp. 914-924.

Barrett, R., 2002. Private communication.

Barrett, R., 1995. Design and manufacturing of adaptive composites for active flight control surfaces. Presented at the Second International Conference on Composites Engineering, New Orleans, LA, August 21-24, 1995.

Bassiouny, E., Ghaler, A.F., and Maugin, G.A., 1988. Thermodynamical formulation for coupled electromechanical hysteresis effects – I. Basic equations. *International Journal of Engineering Science*, **26**, pp. 1279-1295.

Beige, H., and Schmidt, G., 1982. Electromechanical resonances for investigating linear and non-linear properties of dielectrics. *Ferroelectrics*, **41**, 39-39.

Beige, H., 1983. Elastic and dielectric non-linearities of piezoelectric ceramics. *Ferroelectrics*, **51**, 113-119.

Berlington, D., and Krueger, H.H.A., 1959. Domain processes in lead titanate zirconate and barium titanate ceramics. *Journal of Applied Physics*, **30**, 1804-1810.

Bert, C.W., and Birman, V., 1999. Stress dependency of the thermoelastic and piezoelectric coefficients. *AIAA Journal*, **37**, pp. 135-137.

Bert, C.W., and Birman, V., 1998. Effects of stress and electric field on the coefficients of piezoelectric materials: One-Dimensional Formulation. *Mechanics Research Communications*, **25**, pp. 165-169.

Bolotin, V.V., 1964. The Dynamic Stability of Elastic Systems, Holden-Day, San Francisco.

Brown, R.F., and McMahon, G.W., 1962. Material constants of ferroelectric ceramics at high pressure, *Canadian Journal of Physics*, **40**, 672-674.

Brown, R.F., and McMahon, G.W., 1965. Properties of transducer ceramics under maintained planar stress. *Journal of Acoustical Society of America*, **38**, 570-575.

Chaplya, P.M., and Carman, G.P. 2002. Compression of piezoelectric ceramic at constant electric field: energy absorption through non-180 domain-wall motion. *Journal of Applied Physics*, **92**, pp. 1504-1510.

Chattopadhyay, A., Gu, H., and Liu, Q., 1999. Modeling of smart composite box beams with nonlinear induced strain. *Composites: Part B*, **30**, pp. 603-612.

Chen, P.J., and Madsen, M.M., 1981. One dimensional polar responses of the electrooptic ceramic PLZT 7/65/35 due to domain switching. *Acta Mechanica*, **41**, pp. 255-264.

Chen, P.J., and Montgomery, S.T., 1980. A macroscopic theory for existence of the hysteresis and butterfly loops in ferroelectricty. *Ferroelectrics*, **23**, pp. 199-208.

Cho, Y., and Yamanouchi, R., 1987, Non-linear elastic, piezoelectric, electrostrictive and dielectric constants of lithium niobate. *Journal of Applied Physics*, **61**, pp. 875-887.

Fritz, I.J., 1978. Uniaxial stress effects in a 95/5 lead zirconate titanate ceramic. *Journal of Applied Physics*, **49**, pp. 4922-4928.

Ge, P., and Jovuaneh, M., 1995. Modeling hysteresis in piezoceramic actuators. *Precision Engineering* **17**, pp. 211-221.

HaQuang, N., Mook, D.T., and Plaut, R.H., 1987. Non-linear structural vibrations under combined parameteric and external excitations. *Journal of Sound and Vibration*, **118**, pp. 291-306.

HaQuang, N., Mook, D.T., and Plaut, R.H., 1987. A non-linear analysis of the interactions between parametric and external excitations. *Journal of Sound and Vibration*, **118**, pp. 425-439.

Houbolt, J.C., and Brooks, G.W., 1958. Differential equations of motion for combined flapwise bending, chordwise bending, and torsion of twisted nonuniform rotor blades. NACA Report, 1346.

Hsu, C.S., and Cheng, W.-H., 1974. Steady state response of a dynamical system under combined parametric and forcing excitations. *Journal of Applied Mechanics*, **41**, pp. 371-378.

Huang, C.-H., Lin, Y.-C., and Ma, C.-C., 2004. Theoretical analysis and experimental measurement for resonant vibration of piezoceramic circular plates. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, **51**, pp. 12-24.

Joshi, S.P., 1992. Non-linear constitutive relations for piezoceramic materials, *Smart Materials and Structures*, **1**, pp. 80-83.

Krueger, H.H.A., 1954. Mechanical properties of ceramic barium titanate. *Physics Reviews*, **93**, p. 362.

Krueger, H.H.A., 1967. Stress sensitivity of piezoelectric ceramics: Part 1. Sensitivity to compressive stress parallel to the polar axis. *Journal of the Acoustical Society of America*, **42**, pp. 636-645.

Krueger, H.H.A., 1968a, Stress sensitivity of piezoelectric ceramics: Part 2. Heat treatment. *Journal of the Acoustical Society of America*, **43**, pp. 576-582.

Krueger, H.H.A., 1968b, Stress sensitivity of piezoelectric ceramics: Part 3. Sensitivity to compressive stress perpendicular to the polar axis. *Journal of the Acoustical Society of America*, **43**, pp. 583-591.

Leigh, T., and Zimmermann, D., 1991. An implicit method for the non-linear modeling and simulation of piezoceramic actuators displaying hysteresis. *Smart Materials and Structures,* AD-Vol. 24/ AMD-Vol. 123, ASME, ASME Press, New York, pp. 57-63.

Li, L., and Sottos, N.R., 1995. Predictions of static displacements in 1-3 piezocomposites. *Journal of Intelligent Material Systems and Structures*, **6**, pp. 169-180.

Li, L., and Sottos, N.R., 1996a. A design for optimizing the hydrostatic performance of 1-3 piezocomposites. *Ferroelectric Letters*, **21**, pp. 41-46.

Li, L., and Sottos, N.R., 1996b. Measurement of surface displacements in 1-3 and 1-1-3 piezocomposites. *Journal of Applied Physics*, **79**, pp. 1707-1712.

Lian, L., and Sottos, N.R., 1998. Dynamic surface displacement measurement in 1-3 and 1-1-3 piezocomposites. *Journal of Applied Physics*, **84**, pp. 5725-5728.

Maugin, G.A., Pouget, J., Drouot, R., and Collet, B., 1992. Nonlinear Electromechanical Couplings, Wiley, Chichester.

Mettler, E., 1941. Biegeschwingungen eines Stabes mit kleiner Vorkrummung, exzentrisch angreifender publsierender Axiallast und statischer Querbelastung. Forshungshefte aus d. Geb. d. Stahlbaues, **4**, pp. 1-23.

McLachlan, N.W., 1947. Theory and Application of Mathieu Functions, Oxford University Press, New York.

Mukherjee, B.K., Ren, W., Yang, G., Liu, S.F., and Masys, A.J., 2001. Nonlinear properties of piezoelectric coefficients. *Active Materials: Behavior and Mechanics*, SPIE Proceedings, **4333**, Ed. Lynch, S., SPIE, Bellingham, WA, pp. 41-54.

Nakagawa, Y., Yamanouchi, K., and Shibayama, K., 1973, Third-order elastic constants of lithium niobate. *Journal of Applied Physics*, **44**, pp. 3969-3974.

Nguyen, D.V., 1975. Interaction between parametric and forced oscillations in multidimensional systems. *Journal of Technical Physics*, **16**, pp. 213-225.

Nguyen, P.H., and Ginsberg, J.H., 2001. Vibration control using parametric excitation. *ASME Journal of Vibration and Acoustics,* **123**, pp. 359-364.

Priya, S., Viehland, D., Carazo, A.V., Ryu, J., and Uchino, K. 2001. High-power resonant measurements of piezoelectric materials: importance of elastic nonlinearities. *Journal of Applied Physics*, **90**, pp. 1469-1479.

Plaut, R.H., Gentry, J.J., and Mook, D.T., 1990. Non-linear structural vibrations under combined multi-frequency parameteric and external excitations. *Journal of Sound and Vibration*, **140**, pp. 381-390.

Ren, W., Masys, A.J., Yang, G., and Mukherjee, B.K., 2000. The field and frequency dependence of the strain and polarization in piezoelectric and electrostrictive ceramics. Presented at the 3rd Asian Meeting on Ferroelectrics (AMF 3), Hong Kong, December 12-15, 2000.

Sherrit, S., Stimpson, R.B., Wiederick, H.D., and Mukherjee, B.K., 1996. Stress and temperature dependence of the direct piezoelectric charge coefficient in lead zirconate titanate ceramics. *Smart Materials, Structures and MEMS*, Eds. Aarte, V.K., Varadan, V.K., and Varadan, V.V., Proceedings of SPIE, **3321**, pp. 104-113.

Sherrit, S. Wiedenrick, H.D., Mukherjee, B.K., and Sayer, M., 1997. Field dependence of the complex piezoelectric, dielectric, and elastic constants of Motorola PZT 3203 HD ceramic. *Smart Materials Technologies*, SPIE Proceedings, **3040**, Eds. Simmons, W.C. et al., SPIE, Bellingham, WA, pp. 99-109.

Sherrit, S., Catoiu, G., and Mukherjee, K.B., 1999. The characterization and modeling of electrostrictive ceramics for transducers. *Ferroelectrics,* **228**, pp. 167-196.

Sigmund, O., Torquato, S., and Aksay, I.A., 1998. On the design of 1-3 piezocomposites using topology optimization. *Journal of Material Research*, **13**, pp. 1038-1048.

Simitses, G.J., 1986. An Introduction to the Elastic Stability of Structures, Robert Krieger Publishing Company, Malabar, Florida.

Strutt, M.O.J., 1932. Lamesche, Mathieusche and verwandte Functionen in Physik and Technik, Springer, Berlin.

Tan, P., and Tong, L., 2002. A one-dimensional model for non-linear behaviour of piezoelectric composite materials. *Composite Structures*, **58**, pp. 551-561.

Tan, P., and Tong, L., 2001. Micromechanics models for non-linear behavior of piezo-electric fiber reinforced composite materials. *International Journal of Solids and Structures*, **38**, pp. 8999-9032.

Tiersten, H.F., 1971. On the nonlinear equations in thermoelectroelasticity. International *Journal of Engineering Science,* **9**, pp. 587-604.

Tiersten, H.F., 1993a. Equations for the extension and flexure of relatively thin electroelastic plates undergoing large electric fields. *Mechanics of Electromagnetic Materials and Structures*, AMD-Vol. 161/MD-Vol. 42, ASME, New York, pp. 21-34.

Tiersten, H.F., 1993b. Electroelastic equations for electroded thin plates subject to large driving voltages. *Journal of Applied Physics*, **74**, pp. 3389-3393.

Thornburgh, R.P., and Chattopadhyay, A., 2001. Nonlinear actuation of smart composites using a coupled piezoelectric-mechanical model. *Smart Materials and Structures*, **10**, pp. 743-749.

Troger, H., and Hsu, C.S., 1977. Response of a nonlinear system under combined parametric and forcing excitation. *ASME Journal of Applied Mechanics*, **44**, pp. 179-181.

Von Wagner, U., and Hagedorn, P., 2002. Piezo-beam-systems subjected to weak electric field: experiments and modeling of non-linearities. *Journal of Sound and Vibration*, **256**, pp. 861-872.

Von Wagner, U., 2003. Nonlinear longitudinal vibrations of piezoceramics excited by weak electric fields, *International Journal of Non-linear Mechanics*, **38**, pp. 565-574.

Von Wagner, U., and Hagedorn, P., 2003. Nonlinear effects of piezoceramics excited by weak electric fields, *Journal of Nonlinear Dynamics,* **31**, pp. 133-149.

Von Wagner, U., 2004. Non-linear longitudinal vibrations of non-slender piezoelectric rods, *Journal of Non-Linear Mechanics*, **39**, pp. 673-688.

Wang, D.P., and Carman, G.P. 1995. Evaluating the behavior of piezoelectric ceramics subjected to thermal fields. *Adaptive Material Systems*, Summer Symposium of ASME at Los Angeles, AMD-Vol. 206, MD-Vol. 58, pp. 33-47.

Wang, Q-M, Zhang, Q., Xu, B., Liu, R., and Cross, E.L., 1999. Nonlinear piezoelectric behavior of ceramic bending mode actuators under strong electric fields. *Journal of Applied Physics*, **86**, pp. 3352-3361.

Wiederick, H.S., Sherrit, S., Stimpson, R.B., and Mukherjee, B.K. 1996. An optical lever measurement of the piezoelectric charge coefficient. *Ferroelectrics*, **186**, pp. 25-31.

Whitney, J.M., 1987. Structural Analysis of Laminated Anisotropic Plates. Technomic, Lancaster.

Williams, R.B., 2004. Nonlinear mechanical and actuation characterization of piezoceramic fiber composites. PhD thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, March 22, 2004.

Woolett, R.S., and Leblanc, C.L., 1973. Ferroelectric nonlinearities in transducer ceramics. *IEEE Trans. on Sonics and Ultrasonics*, **SU-20**, pp. 24-31.

Yang, G., Liu, S.-F., Ren, W., and Mukherjee, B.K., 2000. Uniaxial stress dependence of the piezoelectric properties of lead zirconate titanate ceramics. *Active Materials: Behavior and Mechanics*, SPIE Proceedings, **3992**, Ed. Lynch, C.S., SPIE, Bellingham, WA, pp. 103-1013.

Appendix: Physically nonlinear effects in Thunder bimorphs used in control surfaces of miniature munitions (development work of Dr. Victor Birman conducted for QorTek, Inc).

The following brief description refers to a program on the development of miniaturized munitions where Dr. Birman conducted the design and development work on bimorph Thunder actuators. While details of the program cannot be disclosed due to the proprietary agreement, some results are shown here.

The Thunder actuators considered in the program represented clamped surfaces with a central substrate bounded by piezoelectric layers that were in turn covered by thin cover sheets (Fig. 16). The electric field applied in the z-direction resulted in the appropriate strains in the longitudinal direction (31 effect). The results for on of two designs considered in the project are shown in Fig. 17. A different length (L) considered in the computations reflected uncertainty about the unsupported length of the cantilevered beam clamped at one end. The curves marked as “Nonlinear” were generated using a physically nonlinear solution. It is obvious that the results shown in Fig. 17, accounting for the effect of physical nonlinearity and using the length of the bimorph equal to 2.9 inches are in excellent agreement with the experimental data.

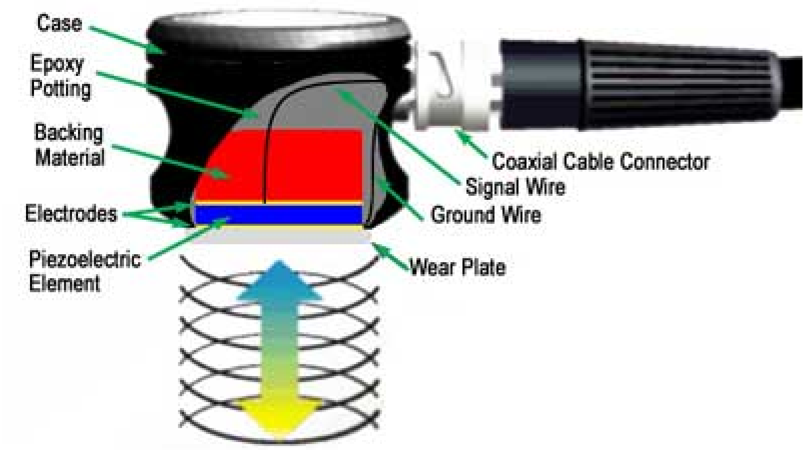

Fig. 1. Piezoelectric transducers used in ultrasonic instrumentation systems (From: http://www.senscomp.com/lseries.htm and http://www.ndt-ed.org/EducationResources/CommunityCollege/Ultrasonics/EquipmentTrans/characteristicspt.htm).

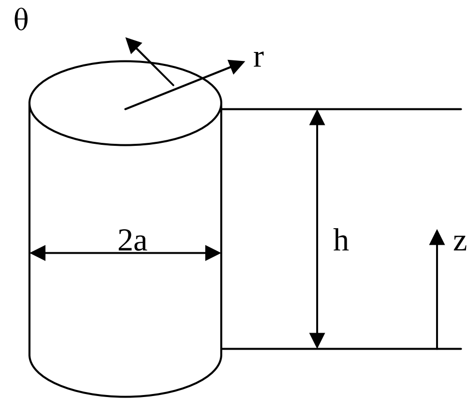

Fig. 2. Cylindrical piezoelectric rod and the coordinate system adopted in the analysis.

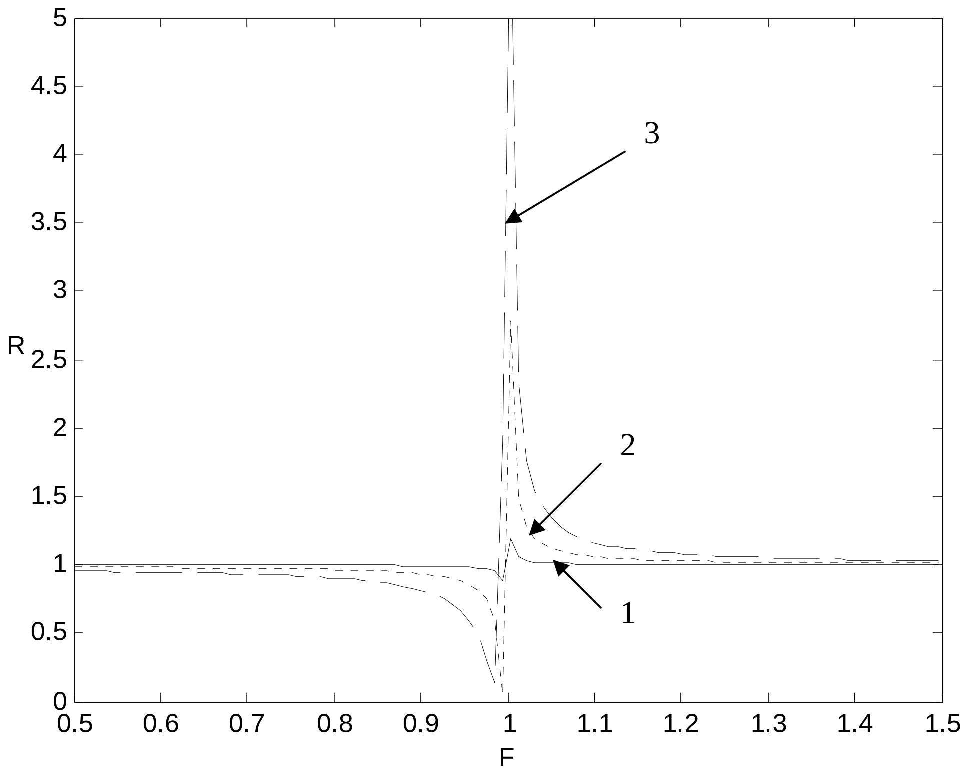

Fig. 3. The ratio of the amplitude obtained without accounting for physical nonlinearity to the physically nonlinear counterpart as a function of the nondimensional frequency for a PZT-5H rod. Damping is neglected. The electric field corresponds to 0.1MV/m, 0.3MV/m and 0.5MV/m for cases 1, 2 and 3, respectively.

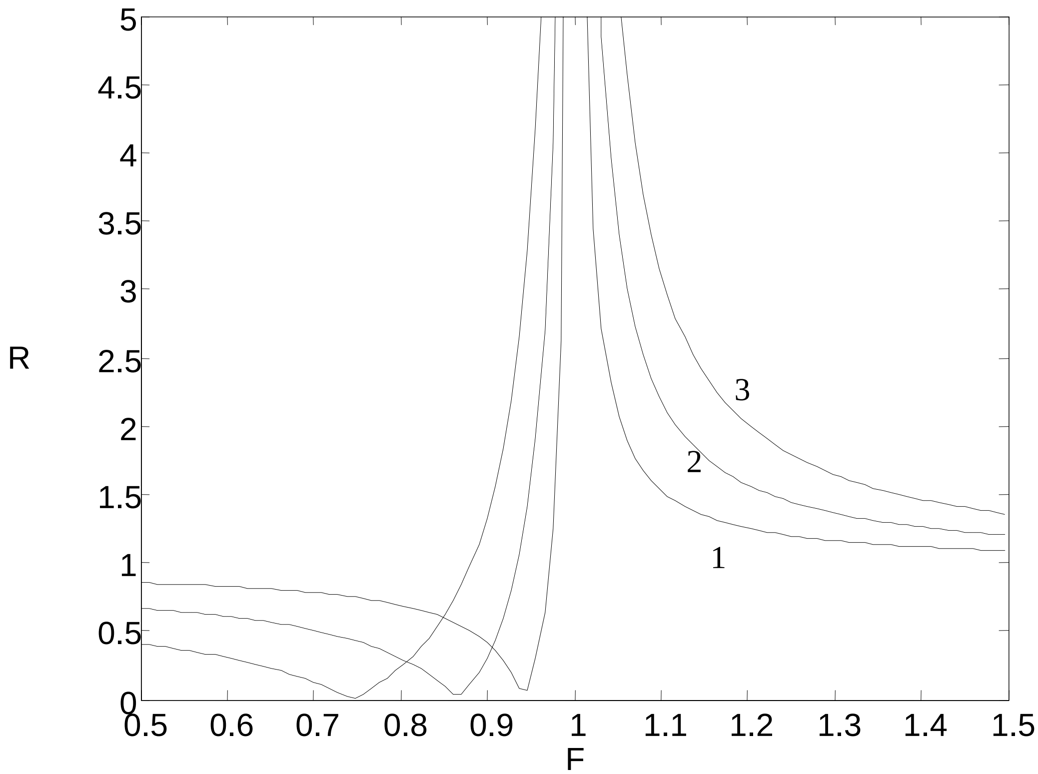

Fig. 4. The ratio of the amplitude obtained without accounting for physical nonlinearity to the physically nonlinear counterpart as a function of the nondimensional frequency for a PZN-4.5%PT rod. Damping is neglected. The electric field corresponds to 1.0MV/m, 1.5MV/m and 2.0MV/m for cases 1, 2 and 3, respectively.

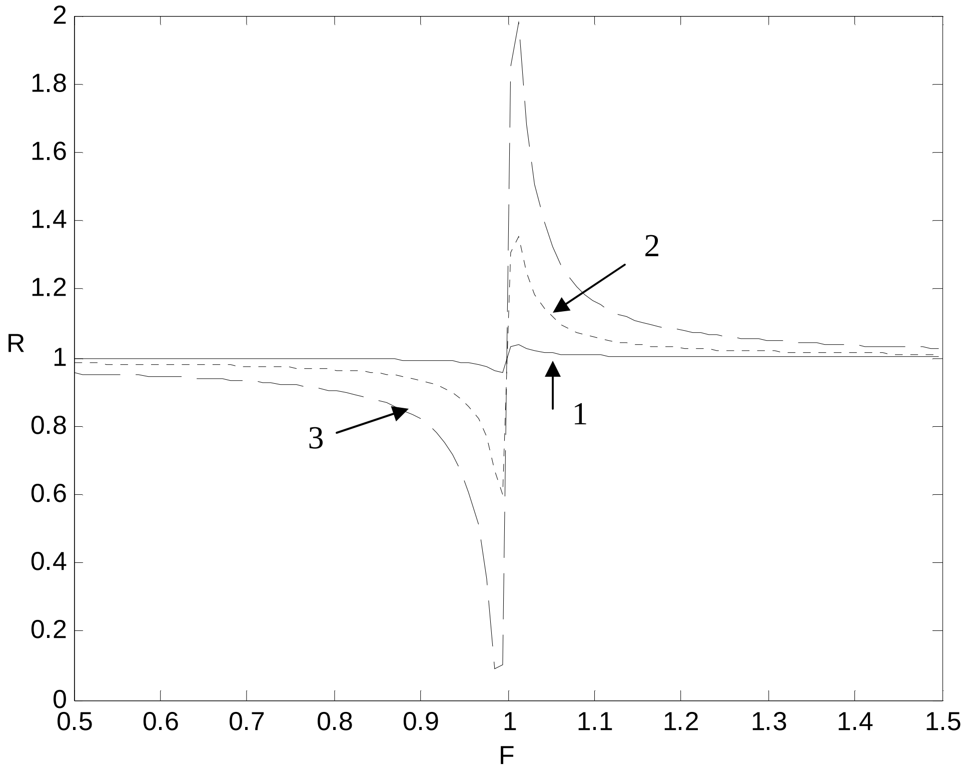

Fig. 5. The ratio of the amplitude of a PZT-5H rod with a physically linear material behavior to the amplitude of the rod accounting for the physically nonlinear effect in the presence of damping. The quality factor is $Q = 65$. The half-depth of the rod is $h = 0.1m$. The electric field corresponds to 0.1MV/m, 0.3MV/m and 0.5MV/m for cases 1, 2 and 3, respectively.

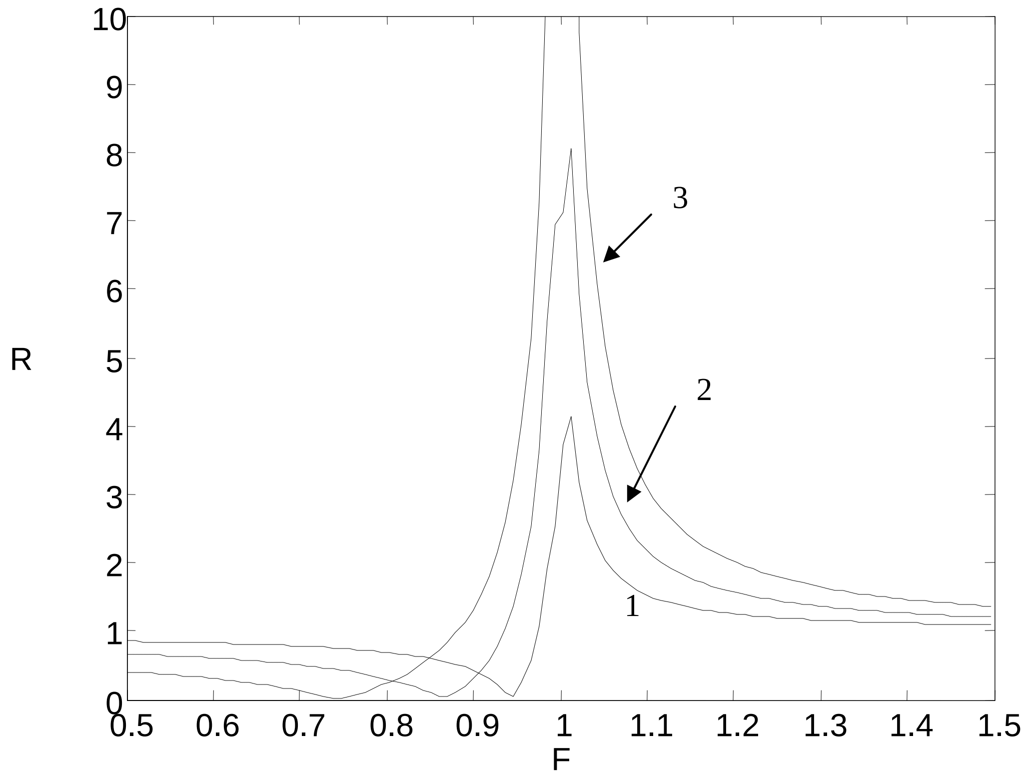

Fig. 6. The ratio of the amplitude of a PZN-4.5%PT rod with a physically linear material behavior to the amplitude of the rod accounting for the physically nonlinear effect in the presence of damping. The quality factor is $Q = 65$. The half-depth of the rod is $h = 0.1m$. The electric field corresponds to 1.0MV/m, 1.5MV/m and 2.0MV/m for cases 1, 2 and 3, respectively.

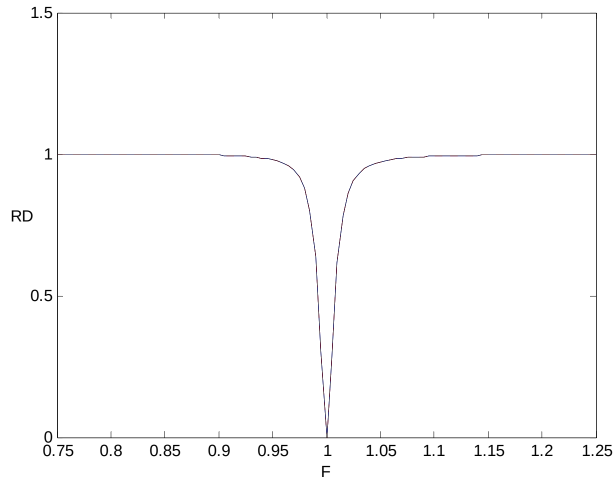

Fig. 7. The ratio of the amplitude of vibrations of a physically nonlinear PZT-5H rod with damping to the amplitude without damping as a function of the nondimensional frequency. The electric field is equal to 0.5MV/m.

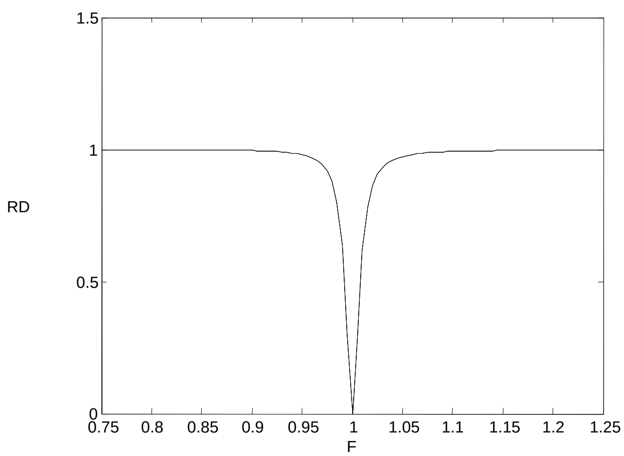

Fig. 8. The ratio of the amplitude of vibrations of a physically nonlinear PZN-4.5%PT rod with damping to the amplitude without damping as a function of the nondimensional frequency. The electric filed is equal to 2.0MV/m.

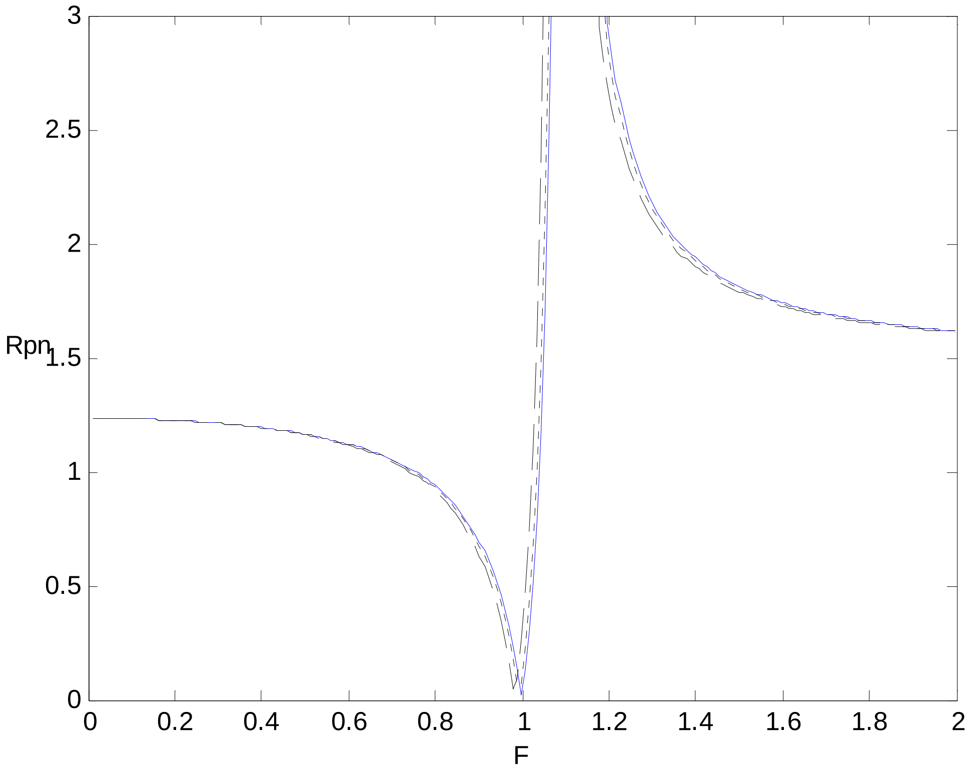

Fig. 9. The ratio of the amplitude of vibrations obtained using power series to the amplitude obtained using normal modes as a function of the nondimensional frequency for a physically nonlinear PZT-5H rod. Three curves corresponding to the electric field equal to 0.1MV/m, 0.3MV/m and 0.5MV/m practically coincide.

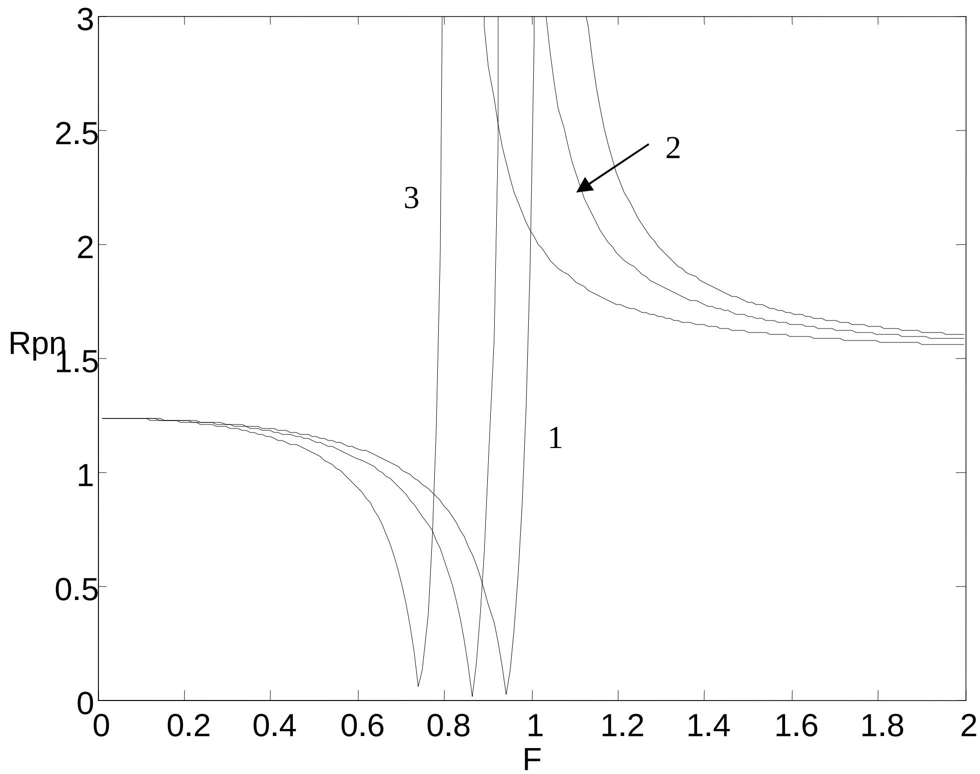

Fig. 10. The ratio of the amplitude of vibrations obtained using power series to the amplitude obtained using normal modes as a function of the nondimensional frequency for a physically nonlinear PZN-4.5%PT rod. The electric field corresponds to 1.0MV/m, 1.5MV/m and 2.0MV/m for cases 1, 2 and 3, respectively.

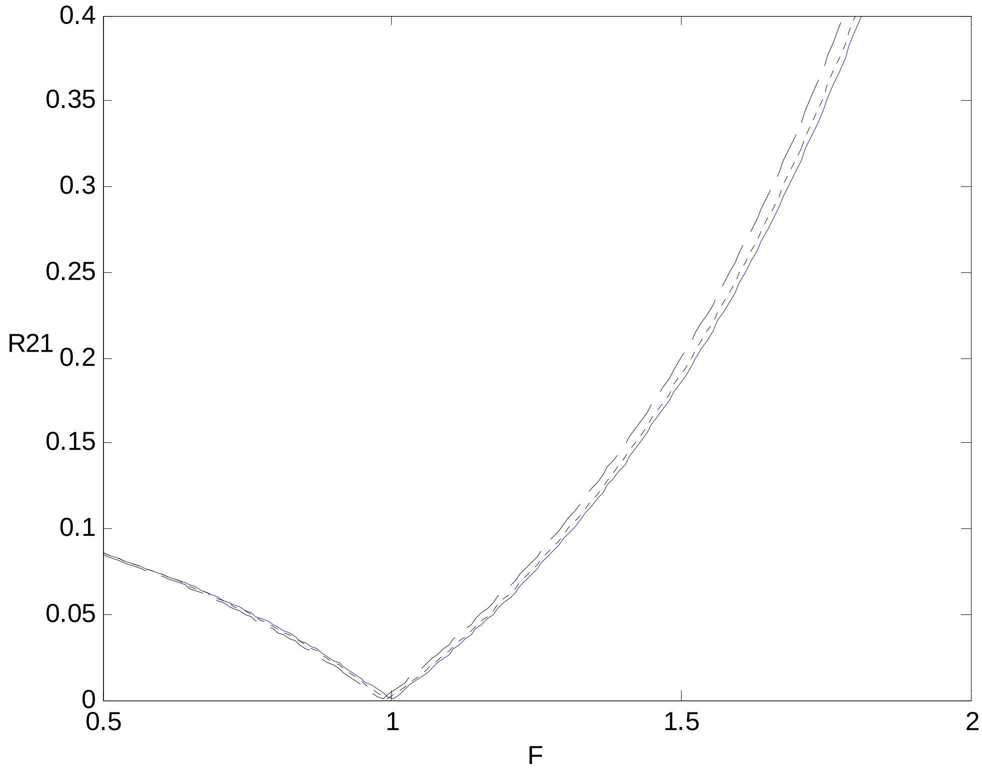

Fig. 11. The ratio of the amplitude of motion corresponding to the second normal mode to that for the first normal mode as a function of the nondimensional frequency for a PZT-5H rod. The curves for the electric fields equal to 0.1MV/m, 0.3MV/m and 0.5MV/m shown by solid, dotted and dashed lines, respectively, practically coincide.

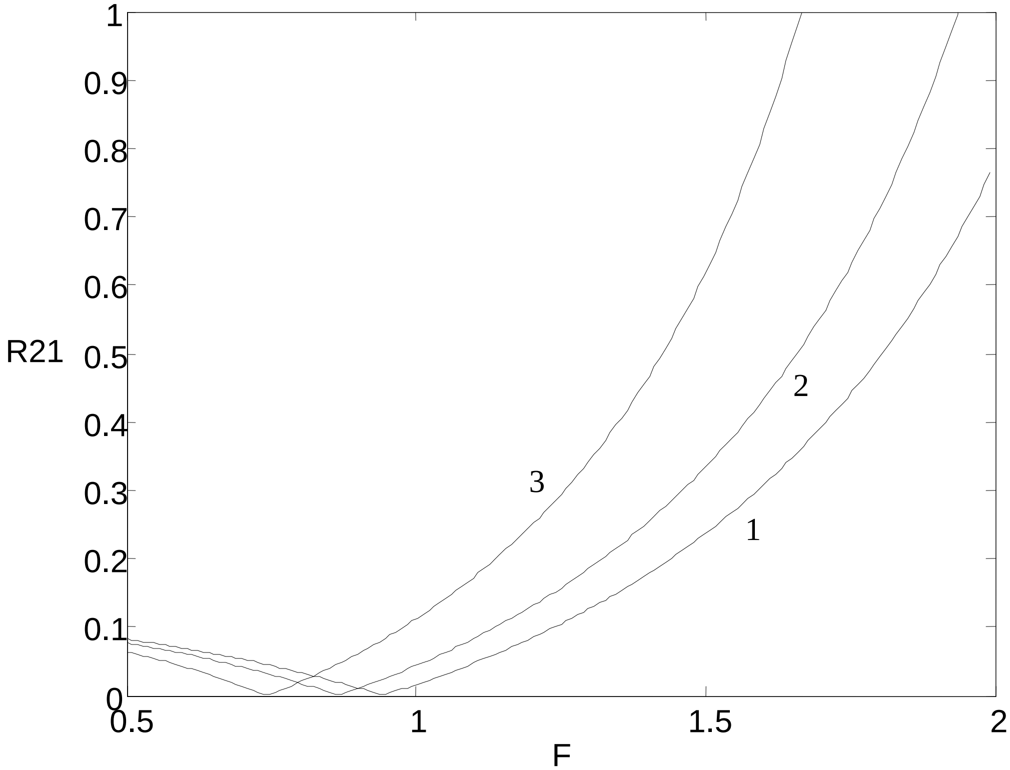

Fig. 12. The ratio of the amplitude of the motion corresponding to the second normal mode to that for the first normal mode as a function of the nondimensional frequency for a PZN-4.5%PT rod. The electric field corresponds to 1.0MV/m, 1.5MV/m and 2.0MV/m for cases 1, 2 and 3, respectively.

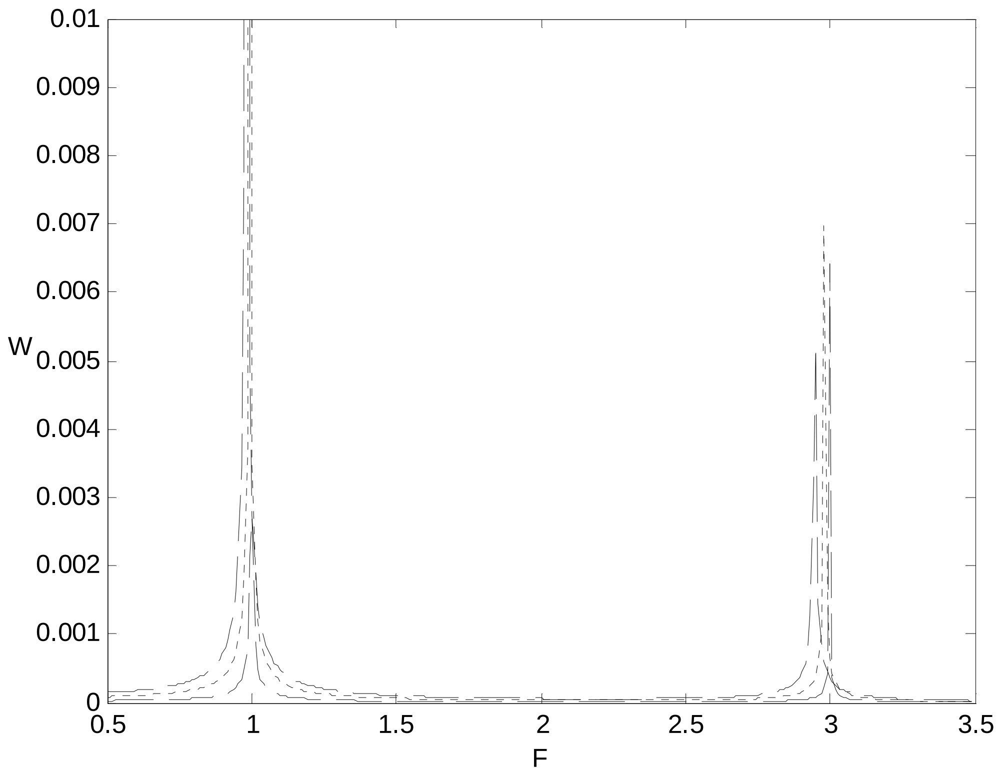

Fig. 13. The nondimensional amplitude of axial vibrations of a PZT-5H rod (h=5mm) as a function of the nondimensional frequency.
Electric field: ———— 0.1MV.m; - - - - - - - - - 0.3MV/m; — — — — — — — 0.5MV/m.

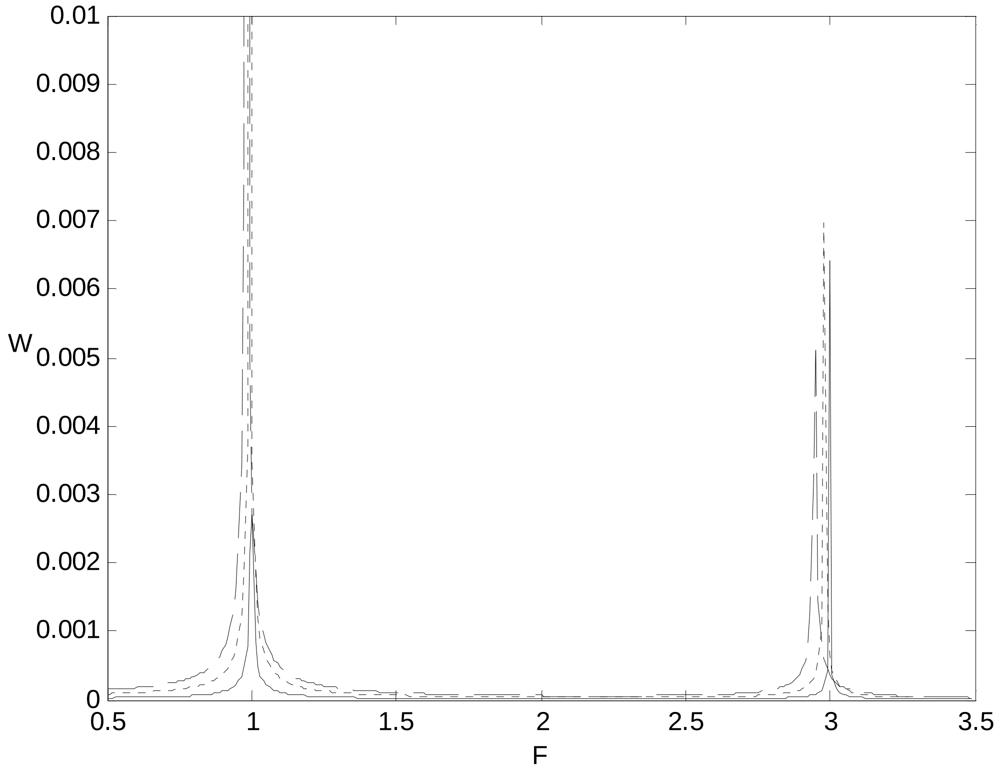

Fig. 14. The nondimensional amplitude of axial vibrations of a PZT-5H rod (h=25mm) as a function of the nondimensional frequency.
Electric field: ———— 0.1MV.m; - - - - - - - - 0.3MV/m; — — — — — — 0.5MV/m.

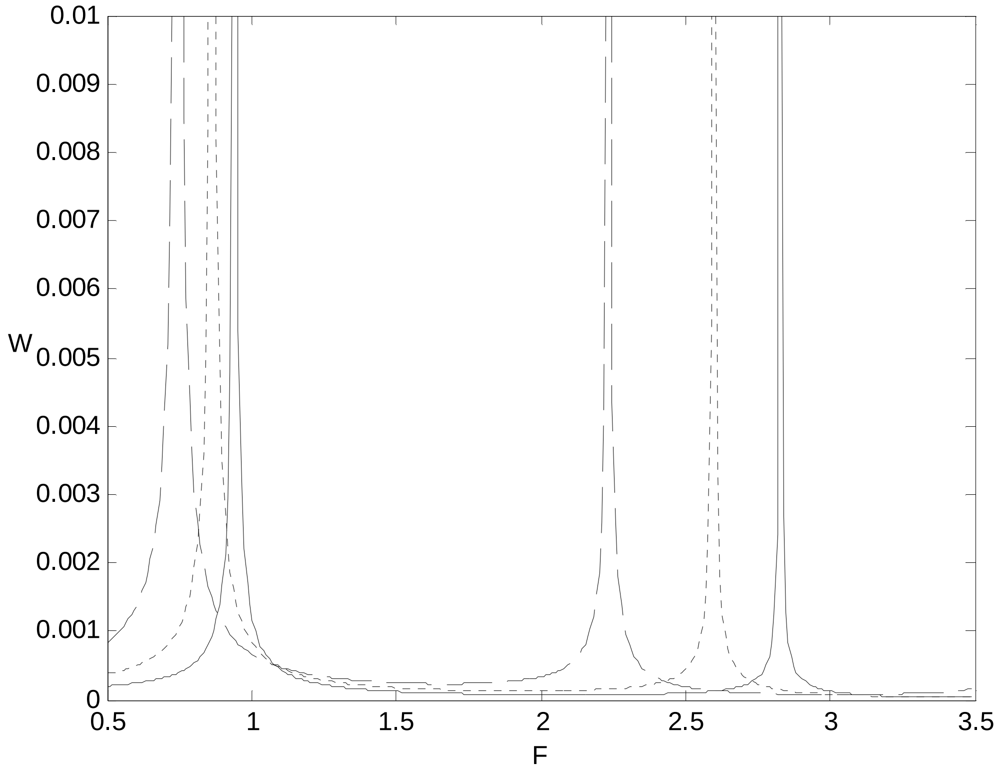

Fig. 15. The nondimensional amplitude of axial vibrations of a PZN-4.5%PT rod (h=25mm) as a function of the nondimensional frequency.
Electric field: ——— 1.0MV.m; - - - - - - - 1.5MV/m; — — — — — 2.0MV/m.

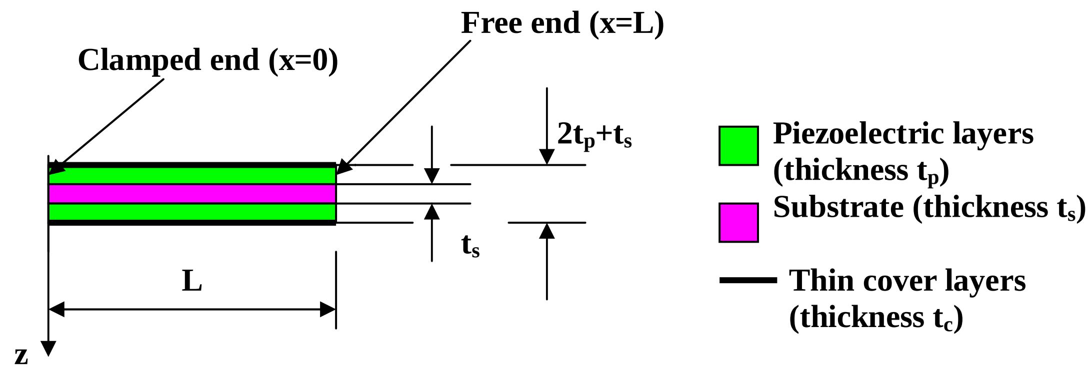

Fig. 16. Cross-section of Model V bimorph (cross section along the axis of the bimorph).

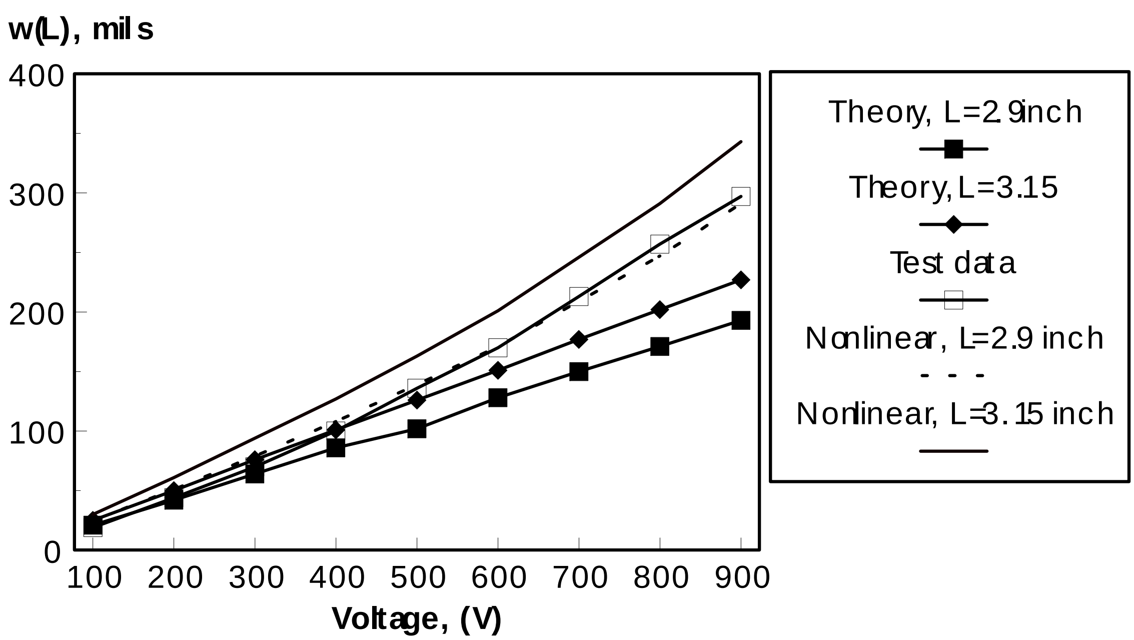

Fig. 17. Comparison between theoretically predicted deflections of the tip of Model V for design 2 (linear and nonlinear results) with experimental data (From Design of Thunder Actuators: Estimate of Deformations and Analysis of a Discrepancy between the Analytical Results and Test Data, Report of Victor Birman, LLC to QorTek, Inc.)

.